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Matroids on the Bases of Simple Matroids

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Let M be a simple matroid (= combinatorial geometry). On the bases of M we consider two matroids $S(M, F)$ and $H(M, F)$, which depend on a field F. $S(M, F)$ is the simplicial matroid with coefficients in F on the bases of M considered as simplices. $H(M, F)$ has been studied by Björner in [1]. It is defined in terms of the order homology of the associated geometric lattice $L(M)$. We prove that $H(M, F)$ is a minor contraction of the full simplicial matroid on all subsets of elements of size $r = r(M)$. Dually this is equivalent to an isomorphism $H(M, F)^* \simeq S(M^*, F)$, where M^* denotes the dual of M. It can be deduced that $H(M, F)$ need not be unimodular, a problem in [1], which inspired this study.

Consider a simple matroid M, i.e. a matroid without loops and parallel elements (cf. [10, p. 51]), sometimes called a combinatorial geometry (cf. [5]). The independent sets of M form a simplicial complex $IN(M)$ the homology of which is determined explicitly in [2], where Björner proves that $\tilde{H}_{r-1}(IN(M))$ is free of rank $\tilde{\mu}(M^*)$ and $\tilde{H}_i(IN(M)) = 0$ for $i \neq r-1$, where r is the rank of M and $\tilde{\mu}(M^*)$ is the Möbius invariant of M^*, the dual of M.

The full simplicial matroid on all r-subsets of a finite set E with coefficients in a field F is denoted by $S_r^E[F]$. A vector representation is given by the mapping

$$A = \{a_1, a_2, \ldots, a_r\} \to \sum_{i=1}^{r} (-1)^{i-1}(a_1, \ldots, \hat{a}_i, \ldots, a_r) = \partial(A) \tag{1}$$

into the chain-group with coefficients in F generated by all oriented $(r-1)$-tuples of elements in E (cf. [3]). Its rank-function is given by $r(X) = |X| - \text{rank } \tilde{H}_{r-1}(X, F)$, where $X \in \binom{E}{r}$ and the homology group is with respect to the simplicial complex generated by X and coefficients in F.

If M is a matroid of rank r on E we may consider the restriction of $S_r^E[F]$ to the set $\mathscr{B} \subseteq \binom{E}{r}$ of bases in M. This simplicial matroid will be denoted by $S(M, F)$. It follows by the above mentioned result of Björner on the homology of $IN(M)$ using the universal coefficient theorem of homology theory

$$\text{rank } S(M, F) = b(M) - \tilde{\mu}(M^*). \tag{2}$$

Let $L = L(M)$ be the geometric lattice associated with M. The supremum operation in L will be denoted by $\vee$. The partial order in L is denoted by $<$. The unique minimal and maximal elements of L are denoted by $\hat{0}$ and $\hat{1}$. The linearly $(<)$-ordered subsets of $L - \{\hat{0}, \hat{1}\}$ give a simplicial complex $\Delta(L)$, the homology of which was determined by Folkman in [7]. Folkman proved that $\tilde{H}_{r-2}(\Delta(L))$ is free of rank $\tilde{\mu}(M) = |\mu(\hat{0}, \hat{1})|$ and $\tilde{H}_i(\Delta(L)) = 0$ for $i \neq r-2$ (reduced homology over the integers).

When $A = \{a_1, a_2, \ldots, a_r\}$ is an r-set of elements in M let

$$\beta(A) = \sum_{\pi} (-1)^{i(\pi)}(a_{\pi(1)}, a_{\pi(1)} \vee a_{\pi(2)}, \ldots, a_{\pi(1)} \vee a_{\pi(2)} \vee \cdots \vee a_{\pi(r-1)}) \tag{3}$$

where the sum is over all permutations $\pi(1), \pi(2), \ldots, \pi(r)$ of the indices $1, 2, \ldots, r$ and $i(\pi)$ is the number of inversions of the permutation.

Björner proved in [1] that $\beta(A)$ is a cycle when A is a base of M and that these cycles generate the non-trivial homology group $\tilde{H}_{r-2}(\Delta(L))$. There is a similar result in [9, Theorem 4.3] by Orlik and Solomon.

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We now define a second matroid $H(M, F)$ on the bases $A \in \mathcal{B}$ by the vector representation $A \to \beta(A)$ with coefficients in F. By the above mentioned results of Folkman, Björner, Orlik and Solomon, it follows using the universal coefficient theorem

$$\operatorname{rank} H(M, F) = \tilde{\mu}(M). \tag{4}$$

THEOREM. *Assume that $M = M(E)$ and M^* are simple matroids. Let $\mathcal{B}$ be the set of bases of M and suppose that rank $M = r$. Then $H(M, F) \simeq S_r^E[F]/(\binom{E}{r} - \mathcal{B})$, where the correspondence is the identity map. Dually $H(M, F)^* \simeq S(M^*, F)$, where the correspondence is $B \leftrightarrow E - B$ when $B \in \mathcal{B}$.*

PROOF. When $A = \{a_1, \ldots, a_{r-1}\} \in \binom{E}{r-1}$ define

$$\sigma(A) = \sum_{\pi} (-1)^{r-1+i(\pi)} (a_{\pi(1)}, a_{\pi(1)} \vee a_{\pi(2)}, \ldots, a_{\pi(1)} \vee \cdots \vee a_{\pi(r-1)}),$$

where the sum is over all permutations π of $1, \ldots, r-1$. It is easy to verify that

$$\sigma(\partial(B)) = \beta(B) \quad \text{when} \quad B \in \binom{E}{r}. \tag{5}$$

By the lemma of Orlik and Solomon [9, Lemma 3.8] is

$$\beta(B) = 0 \text{ when } B \text{ is dependent of size } r \text{ in } M. \tag{6}$$

Let V_1, V_2, V_3 be the vector spaces over the field F generated respectively by $\{\partial(B) : B \in \binom{E}{r}\}$, $\{\partial(B) : B \in \binom{E}{r} - \mathcal{B}\}$, $\{\partial(B) : B \in \mathcal{B}\}$.

The map $\sigma : V_1 \to V_3$ is linear. Since $\sigma(V_2) = 0$ by (6), we have the induced linear map $\bar{\sigma} : V_1/V_2 \to V_3$. It follows that the identity is a strong map of $S_r^E[F]/(\binom{E}{r} - \mathcal{B})$ onto $H(M, F)$. The isomorphism will follow from

$$\operatorname{rank} S_r^E[F] \Big/ \left(\binom{E}{r} - \mathcal{B}\right) = \operatorname{rank} H(M, F) \tag{7}$$

by a well known theorem on strong maps.

By "Tutte's law" $(M/A)^* = M^* - A$ and since $(S_r^E)^* \simeq S_{|E|-r}^E$, we have

$$S_r^E[F] \Big/ \left(\binom{E}{r} - \mathcal{B}\right) \simeq (S_{|E|-r}^E[F](\mathcal{B}'))^* = S(M^*, F)^*, \tag{8}$$

where $\mathcal{B}' = \{E - B : B \in \mathcal{B}\}$. The relation (7) follows by (2), (4) and (8). Therefore the first isomorphism in our theorem is true. The second isomorphism then follows by (8).

EXAMPLE. Let M be the uniform matroid of rank r and $|E| = n$, U_r^n, $2 \le r \le n-2$. Then $S(U_r^n, F) = S_r^n[F]$. By our theorem it follows $H(U_r^n, F) \simeq S_r^n[F]$, which is unimodular (= regular) if and only if $k = 2$ or $n - 2$ (cf. [4, 8]).

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REFERENCES

1. A. Björner, On the homology of geometric lattices, *Algebra Universalis*. to appear.
2. A. Björner, Homology of matroids (to appear in [6]).

3. R. Cordovil, Sur les géométries simpliciales, *C.R. Acad. Sci. Paris Sér. A* **286** (1978), 1219–1222.
4. R. Cordovil and M. Las Vergnas, Géométries simpliciales unimodulaires, *Discrete Math.* **26** (1979), 213–217.
5. H. Crapo and G.-C. Rota, *Combinatorial Geometries*, M.I.T. Press, Cambridge, Massachusetts, 1970.
6. H. Crapo, G.-C. Rota and N. White, *Combinatorial Geometries: the Theory of Matroids*, to appear.
7. J. Folkman, The homology groups of a lattice, *J. Math. Mech.* **15** (1966), 631–636.
8. B. Lindström, Non-regular simplicial matroids, *Discrete Math.* **28** (1979), 101–102.
9. P. Orlik and L. Solomon, Combinatorics and topology of complements of hyperplanes, *Invent. Math.* **56** (1980), 167–189.
10. D. J. A. Welsh, *Matroid Theory*, Academic Press, London, 1976.

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